It is a very simple matter to state numerous other criterions for the symmetry of polynomials of n variables. In Netto's *Theory of Substitutions*, 1892, page 90, it is stated that the probability that any two substitutions on n letters generate the symmetric group may be taken as about $\frac{3}{4}$. That is, if we interchange the n variables of a polynomial according to two substitutions, and if the polynomial remains unchanged, the probability that it is symmetric is about $\frac{3}{4}$. Although this implies that numerous tests for symmetry may readily be obtained, it is doubtful whether it is possible to find more useful general tests than those given in the preceding paragraphs. Among the most convenient additional tests is the following: *A necessary and sufficient condition for a polynomial in n variables to be symmetric is that it be unchanged by each of the* $n-1$ *interchanges which result when we interchange the first and the second variable, then the second and third, then the third and fourth, and finally the* $(n-1)$*th and the nth.*

Although the last theorem requires $n-1$ tests to prove the symmetry of a polynomial, each of these tests is so very simple that the total number of them involve about the same amount of labor as each of the sets of two tests given in the preceding theorems. In this connection it may be observed that the symmetric group of degree n can always be generated by a pair of substitutions of orders 2 and 3 respectively, except when n is one of the three numbers 5, 6, 8.* Hence we may prove the symmetry of a polynomial of any degree, besides these three, by means of two substitutions of orders 2 and 3 respectively. In practice this kind of general test is, however, less simple than some of those given above. On the other hand, the noted Italian mathematician, Alfredo Capelli, gave a test which is fairly convenient and can be easily stated as follows: *A necessary and sufficient condition for a polynomial to be symmetric is that it be unchanged by a cyclic interchange of all its variables as well as by the cyclic interchange of all but one of its variables, the variables having the same relative positions in the two cycles.*†

Deep interest in the symmetric functions was first aroused by the fact that the coefficients of the general equation of degree n, are symmetric functions of the roots. In 1770, Lagrange and others began the study of rational integral functions of n variables which are not necessarily symmetric, and observed that the number of different formal values assumed by such functions, when the variables are permuted in every possible way, is always a division of $n!$ Just as the permutations which do not affect the formal value of a symmetric polynomial, correspond to a substitution group, so there is also connected with each non-symmetric polynomial a substitution group which sheds light on its properties. In particular the number of different formal values of the polynomial, when its variables are permuted in every possible manner, is equal to the index of this substitution group, and the possibility of constructing polynomials with a given number of

* *Bulletin of the American Mathematical Society*, Vol. 7 (1901), p. 426.
† Capelli, *Giornale di Matematiche*, Vol. 35 (1897), p. 354.

formal values, is completely determined by the possible substitution groups on these variables. The above remarks apply directly to all rational functions of the n variables as well as to the more special functions which are commonly (but not universally) called polynomials.

ON THE COMBINATION OF INVOLUTIONS.*

By D. N. LEHMER, University of California.

1. Given two involutions of rays in the same plane with centers at A and B a quadratic reciprocal transformation may be set up in the plane as follows: To any point P make correspond the intersection P' of the two rays at A and B which correspond in the involutions to the rays PA and PB. The point-to-point transformation thus defined is clearly involutorial. Certain points appear as exceptional in that the correspondence is not unique, namely: all of the points on the line joining A and B go by the transformation into the same point C. Further, all of the points on AC go into the same point B, and all of the points on BC go into the point A, as is seen by making the construction according to the definition. It will be suspected that the point C is the center of an involution which might be used instead of A or B to define the same transformation. That this is the case appears as a corollary from the following fundamental theorem.

2. Theorem. *If the point P describes a straight line the corresponding point P' describes a conic through A, B, and C.*

The point row P projects to A and B in two perspective pencils. The corresponding rays in the involutions at A and B are therefore projective and generate a conic through A and B. Since the point row P meets the line AB in one point the corresponding point P' goes through C. From the construction it appears further that the point row of the first order P is projective to the point row of the second order P'. It follows therefore that the pencils PC and $P'C$ are projective and that they are in involution. The point C is thus in all respects coördinate with A and B.

If we assume the theorem that a conic meets a curve of degree m in $2m$ points we may prove easily the more general theorem.

3. Theorem. *A curve of degree n goes by the above transformation into a curve of degree $2n$.*

For let the curve C of degree n go into a curve C'. Cut across C' by a line g. Transform C' and g. C' goes back into C and g goes into a conic γ. The points common to C and the conic γ, $2n$ in number, correspond to the points common to C' and the arbitrary line g.

* Presented at the September meeting of the San Francisco Section of the American Mathematical Society, under slightly different title.

4. In the above theorems we have assumed tacitly that the curve described by P does not pass through A, B, or C. If for instance in the first theorem the line traversed by P passes through A, the point P' must traverse the line corresponding to it in the involution at A. The conic in fact degenerates into this line and the line BC which corresponds to the point A. The more general theorem of paragraph 3 should be stated as follows to take account of this degeneracy. A curve of degree n which passes k times through the points A, B, or C goes into a degenerate curve of degree $2n$, which contains the sides of the triangle counted k times and a curve of degree $2n-k$. Thus a conic goes generally into a quartic which has A, B, and C for double points since the conic cuts the sides of the triangle generally in two points. If the conic goes through the point A the corresponding curve is a cubic with double point at A. If the conic contains two points A and B the corresponding curve is a conic through A, B, and C. If the conic passes through all three points the corresponding curve is a straight line.

5. The theory of the quadratic transformation is of course perfectly well known and many ways have been devised for realizing it geometrically.* The above method appears in many ways the simplest and most effective. Thus the four invariant points appear as the intersections of the focal rays at A with the focal rays at B; and since the focal rays at C must go through them they are presented as the vertices of a quadrilateral of which ABC is the diagonal triangle. Again, since AB and AC are corresponding rays in the involution at A that involution is determined by one pair of corresponding points P, P'. Two transformations S and T having the same base triangle ABC produce a collineation in the plane. For S throws a line into a conic through ABC, and T throws this conic into a line again. Conversely, it is easy to show that any collineation is the product in this way of two quadratic transformations. The vertices A, B, C, are the self-corresponding points in the collineation.

6. We have seen that two involutions A and B determine a third C. We will consider now three involutions A, B, and C which are not so related. It is clear that a point P in the plane will not generally have a point P' conjugate to it with respect to all three involutions. For the rays at A, B, and C which correspond to PA, PB, and PC will not generally meet in a point. Points may be found, however, which do have conjugates with respect to all three as the following theorem shows.

7. Theorem. *On any line in the plane there is at least one point and at most three which have conjugate points with respect to three unrelated involutions.*

For let the point P traverse the line g. The rays at A and B which correspond in the involutions at those points to PA and PB generate as we have seen a conic through A and B. There is also a conic through A and C. These two conics have a point A in common, and so have at least one and at

* See Reye, *Geometry of Position*, Holgate's Translation, p. 111.

most three other common points. These are easily seen to be points of the kind in question. Thus we have the following

Theorem. *The locus of points which have conjugate points with respect to three unrelated involutions in the plane is a cubic curve.*

8. If a point P is on the curve the point P' is also, as appears from the character of the correspondence. The points of the curve are thus paired in involution in the sense that four points P that project to A, B, or C in four harmonic rays correspond to four points P' that also project to A, B, or C in four harmonic rays. The points A, B, and C are easily seen to be points on the curve. To show this, construct the point of intersection C' of the rays at A and B which correspond to CA and CB. C and C' are conjugate points.

9. *If PP' and QQ' are two pairs of conjugate points so also are the points $(PQ, P'Q')=R$ and $(PQ', P'Q)=R'$.* For joining PP' and QQ' to A we get two pairs of corresponding rays in the involution at A. But if we project to A the opposite vertices of the complete quadrilateral PP', QQ', RR', we get three pairs of rays in the same involution. Thus the vertices RR' project to A and similarly to B and C in the involutions at those points.

10. *If P and P' are fixed and Q is taken arbitrarily on the curve, then QP and QP' meet the curve again in conjugate points R and R'.*

If Q moves so as to make R approach P then the conjugate R' must approach P' and we have at once the beautiful

Theorem. *The tangents to the curve at a pair of conjugate points meet again on the curve.*

In this case Q' lies on the line joining PP'.

11. Consider now four unrelated involutions of rays in the same plane. Is it possible to find points in the plane conjugate with respect to all four? Let the centers be A, B, C, and D. Points conjugate with respect to A, B, and C lie on a cubic through those points. Points conjugate with respect to A, B, and D lie on a cubic through those points. The two cubics have A and B for common points and also the point M, where M is the center of the third involution equivalent to, and determined by, the involutions at A and B, as in the first paragraph. Other intersections must occur in pairs, for if P lies on both cubics so also must P', which is conjugate to it with respect to all four involutions. If they have two pairs of points in common they must have a third pair. If they have still another pair it is possible to find as many pairs as one desires by the above quadrilateral construction. In that case the curves coincide and D lies on the cubic determined by A, B, and C. It follows that *the cubic may be generated by any three points on it as centers of involution*, provided, however, the three points are not centers of related involutions.

It follows from this that *any point on the cubic is joined to three pairs of conjugate points by three pairs of rays in involution.*

12. This last theorem is indeed the fundamental starting point for the

usual treatment of the general cubic by synthetic methods.*

13. We have omitted to speak of the corresponding discussion that might be made of point rows in involution. The extension of the method to space of three dimensions is of interest, however. Consider three quadric surfaces, A, B, and C. Then to any point in space P, we make correspond the point P' of intersection of the polar planes of P with respect to the three surfaces.† The correspondence is reciprocal and generally unique. By it *a straight line goes into a twisted cubic.* The polar planes of P describe projective axial pencils. If we cut across by an arbitrary plane we get three projective pencils of rays in that plane. By a discussion precisely similar to the discussion in paragragh 7 it is shown that one set of corresponding rays at least and three sets at most will meet in a point. These are points on the locus of P'.

14. By the above transformation *a plane goes into a cubic surface.* For suppose the plane μ goes into the surface M. Cut across M by an arbitrary line g. Reciprocate and the surface M goes into the plane μ and g goes into a twisted cubic which meets μ in one or three points. These are conjugate to the three points in which g meets M.

15. By an entirely similar argument, assuming the theorem: *A twisted cubic meets a surface of degree n in $3n$ points,* we may show that, *A surface of degree n goes into a surface of degree $3n$.* Also by assuming that, *A cubic surface intersects a space curve of degree n in $3n$ points,* we may show that, *The above transformation throws a curve of degree n into a curve of degree $3n$.*

16. We have seen in the corresponding plane theory that a straight line is transformed into a conic, and a conic into a quartic. The particular conic that corresponds to a straight line however transforms back not into a quartic but into the line. This was explained by the fact that the conic obtained from a straight line passed through each of the vertices of the triangle ABC and thus degenerated in the transformation into a quartic made up of four lines. A similar difficulty appears in the space theory. A plane γ goes into a cubic C, and this cubic C instead of going into a surface of degree nine, as the theory requires, goes back into the plane γ again. Also the cubic curve that corresponds to a straight line goes back into a straight line again and not into a curve of degree nine. To explain this difficulty, consider the plane γ and the corresponding cubic surface C_3. Cut across C_3 by any plane α. Reciprocate C_3 and α. C_3 goes into a plane γ, and α into a cubic surface A. Therefore the curve of intersection of α with C_3 goes into a cubic curve C lying in γ. Cut C_3 by a second plane β, and let the intersection line of α and β meet C_3 in three points P, Q, and R. As before, the curve of intersection of β and C_3 goes into a cubic curve b in γ. The two curves b and c have the three points P', Q', and R' in common. They have then generally six others, S_1, S_2, S_3, S_4, S_5, S_6. It is seen that S_1 corresponds to a point on

* See Schraeter, *Ebene Kurven Dritter Ordnung.* Leipzig, 1888.

† This correspondence is due to Steiner, *Gess. Werke, II,* p. 651.

α since it is a point on a, and also to a point on β since it is a point on b. But this means that the polar planes of S_1 meet in two distinct points and therefore all three polar planes meet not in a single point but in a line. There are six such points generally in any plane. *The locus of points therefore, such that their polar planes with respect to three quadric surfaces meet in a line, is a twisted sextic curve.* It is of course the Jacobian of the three surfaces.*

17. We note in passing that the six lines corresponding to the six points S in γ lie entirely in the surface C_3. Also that just as in the plane theory if a line goes through one of these points the corresponding twisted cubic degenerates into a conic and a straight line. Also if a line goes through two of these points the corresponding twisted cubic is made up of three straight lines. To the fifteen lines therefore joining the six points S in pairs, correspond fifteen other lines lying in the surface. Further, the conics through any five correspond to degenerate curves of degree six made up of six straight lines, five of which are the lines corresponding to the points S. The sixth also, therefore, lies on the surface giving five more or twenty-seven in all. The usual theorems may of course be made to follow.†

18. To a point on the sextic curve corresponds thus a line, and to any point on that line corresponds back again the point on the sextic. If then, a point P move along one of these lines the corresponding three axial pencils described by its polar planes have their axes meeting in the point on the sextic. The argument of paragraph 13 can be extended to show that if the axes of three projective pencils meet in a point there will be three sets of corresponding planes at most and one set at least that meet in the same straight line. This means that there are three positions for P on the line in question in which its three polar planes meet in a line. In other words, *Every line corresponding to a point on the sextic curve meets the sextic in one or three points.* From the proof we have established also that, *Through every point on the sextic curve will pass one or three lines of the sort in question.* It is of interest then to investigate the singly infinite system of lines which correspond to points on the Jacobian sextic of three quadrics.

19. To obtain the degree of the surface O of which these lines are the generators we observe first that a straight line must meet it in eight points. For the twisted cubic corresponding to the line must cut the Jacobian sextic eight times. Otherwise the twisted cubic would go back into a curve of degree higher than one. The curve of degree nine into which this twisted cubic must go is thus a degenerate curve made up of the original line and eight lines of the surface O. Again a plane goes into a cubic surface. This cubic surface contains every point of the sextic since the plane from which it is derived cuts every line of the surface O. The cubic surface goes back into a plane instead of into a surface of degree nine. This means that the surface O is of the eighth degree.

* See Salmon's *Geometry of Three Dimensions*, p. 213.

† Reye, l. c., Chapter 26.

20. This surface O is thus a ruled surface of degree eight, every ruling of which meets the Jacobian in three points (or one) and such that three (or one) rulings go through every point on the sextic. The Jacobian is thus a triple line on the surface, three sheets of the surface passing through it. A plane section of the surface will thus give a curve of degree eight with six triple points.

21. This surface O and the Jacobian sextic of the three quadric surfaces will serve to explain the discrepancies in the degrees of reciprocal curves and surfaces. Thus a curve of degree n goes by the transfarmation into a curve of degree $3n$. But since the curve of degree n meets the surface O in $8n$ points, the curve of degree $3n$ meets the sextic in $8n$ points. When therefore, we reciprocate *this* curve of degree $3n$ part of the resulting curve is made up of $8n$ lines of the surface O, and the degree of the resultant curve is thus $9n-8n$ or n, which is the original curve. Similarly for surfaces. Every ruling of O meets a surface of degree n in n points, so that the reciprocal surface of degree $3n$ passes n times through the Jacobian curve. In reciprocating the surface back again we get a surface of degree $9n$ to be sure, but the surface O is n times a part of it. The reciprocal surface is thus of degree $9n-8n$ or n, the original surface.

PROOF OF THE FIRST FORMULA FOR EVALUATING 0/0.

By N. J. LENNES, Columbia University, New York.

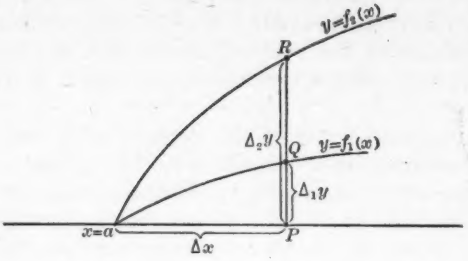
It is well known that many proofs of theorems, while entirely logical and conclusive, afford little or no insight into the theorems proved and certainly furnish no obvious clue to the origin either of the theorem or the proof. On the other hand it is by no means infrequent that one has direct insight into the nature of a theorem or mathematical process and that such insight may be of great value both as an aid to the memory and in judging the true value and uses of the theorem or process while at the same time it may be found impossible, or at least impracticable, to formulate the elements of thought which constitute this insight into a logical proof.

It is believed to be a pedagogical principle of wide application that, other things being equal, the one of two modes of treatment of any topic is to be preferred which affords the more direct and obvious connection between intuitional insight on the one hand and rigorous logical proof on the other. For this reason the proofs given below seem to be worth publishing.

Theorem. *If* $f_1(x)$ *and* $f_2(x)$ *are continuous functions each possessing a derivative at* $x=a$, *and such that* $f_1(a)=0$, $f_2(a)=0$, *while* $f_2'(a)\neq 0$, *then*

$$\operatorname{Lim}_{x=a} \frac{f_1(x)}{f_2(x)} = \frac{f_1'(a)}{f_2'(a)}.$$

Proof. Using the notation indicated in the figure,



$$f_1'(a)=\underset{\Delta x \doteq 0}{\text{Lim.}}\frac{\Delta_1 y}{\Delta x}$$

and $$f_2'(a)=\underset{\Delta x \doteq 0}{\text{Lim.}}\frac{\Delta_2 y}{\Delta x},$$

and hence $$\frac{f_1'(a)}{f_2'(a)}$$

$$=\underset{\Delta x \doteq 0}{\text{Lim.}}\frac{\frac{\Delta_1 y}{\Delta x}}{\frac{\Delta_2 y}{\Delta x}}. \quad \text{But} \quad \frac{\frac{\Delta_1 y}{\Delta x}}{\frac{\Delta_2 y}{\Delta x}}=\frac{\Delta_1 y}{\Delta_2 y}=\frac{f_1(x)}{f_2(x)}.$$

$$\text{Hence,}\quad \frac{f_1'(a)}{f_2'(a)}=\underset{x \doteq a}{\text{Lim.}}\frac{f_1(x)}{f_2(x)}.$$

In case $f_2'(a)=0$ and also $f_1'(a)=0$, the more general theorem is established in the usual manner as follows:*

By the generalized mean value theorem; namely, that if $f_1(x)$ and $f_2(x)$ have derivatives on the interval ab, then there is a value of x, X, on ab, such that

$$\frac{f_1(a)-f_1(b)}{f_2(a)-f_2(b)}=\frac{f_1'(X)}{f_2'(X)}.$$

Since in the case of our theorem we have $f_1(a)=0$ and $f_2(a)=0$ it follows that in the above figure

$$\frac{f_1(b)}{f_2(b)}=\frac{f_1'(X)}{f_2'(X)},$$

or, what is the same thing,

$$\frac{f_1(x)}{f_2(x)}=\frac{f_1'(X)}{f_2'(X)},$$

* See, for instance, Osgood's *Calculus*, pp. 234, 235.

where X is some point between a and x. Hence, obviously,

$$\operatorname{Lim}_{x=a} \frac{f_1(x)}{f_2(x)} = \operatorname{Lim}_{x=a} \frac{f_1'(X)}{f_2'(X)}.$$

If we now assume $f_2''(a) \neq 0$, we have, by the theorem first proved,

$$\operatorname{Lim}_{x=a} \frac{f_1(x)}{f_2(x)} = \operatorname{Lim}_{x=a} \frac{f_1'(x)}{f_2'(x)} = \frac{f_1''(a)}{f_2''(a)}.$$

Obviously, by the same process, in case $f_2''(a) = 0$, but $f_2'''(a) \neq 0$,

$$\operatorname{Lim}_{x=a} \frac{f_1(x)}{f_2(x)} = \frac{f_1'''(a)}{f_2'''(a)},$$

and so on for the higher cases.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

348. Proposed by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

Prove the following relation between Jacobi's symbols:

$$\left(\frac{d}{n}\right) = (-1)^{m(d-1)/2}\left(\frac{d}{2md-n}\right)$$

where d and n are positive odd numbers and $2md > n$.

I. Solution by the PROPOSER.

Replace each symbol by the value given by the generalized reciprocity theorem; then on the right apply

$$\left(\frac{2md-n}{d}\right) = \left(\frac{-n}{d}\right) = (-1)^{(d-1)/2}\left(\frac{n}{d}\right).$$

Hence the relation will be true if

$$\frac{n-1}{2}\cdot\frac{d-1}{2}\equiv\frac{m\,(d-1)}{2}+\frac{d-1}{2}\cdot\frac{2md-n-1}{2}+\frac{d-1}{2}\pmod 2,$$

or,

$$\frac{(d-1)}{2}[m+md-n+1]\equiv 0\pmod 2.$$

But the first factor is an integer, and the second is an even integer.

II. Solution by BENJAMIN FRANKLIN YANNEY, A. M., Mount Union College, Alliance, Ohio.

An implied assumption is that d and n are relatively prime to each other. It follows, then, that d and $2md—n$ are also relatively prime to each other.

1. By the reciprocity theorem, $\left(\frac{d}{n}\right)\left(\frac{n}{d}\right)=(-1)^{\frac{1}{2}(d-1)\cdot\frac{1}{2}(n-1)}$.

2. Multiply by $\left(\frac{-1}{d}\right)=(-1)^{\frac{1}{2}(d-1)}$, member by member:

$$\left(\frac{d}{n}\right)\left(\frac{n}{d}\right)\left(\frac{-1}{d}\right)=(-1)^{\frac{1}{2}(d-1)\cdot\frac{1}{2}(n-1)+\frac{1}{2}(d-1)}=(-1)^{\frac{1}{2}(d-1)\cdot\frac{1}{2}(n+1)}.$$

But, $\left(\frac{n}{d}\right)\left(\frac{-1}{d}\right)=\left(\frac{-n}{d}\right)=\left(\frac{2md-n}{d}\right)$, as is well known. [See (35) and (36), p. 60, Bachmann's *Neuere Zahlentheorie.*]

3. Therefore, $\left(\frac{d}{n}\right)\left(\frac{2md-n}{d}\right)=(-1)^{\frac{1}{2}(d-1)\cdot\frac{1}{2}(n+1)}$.

4. Multiply by $\left(\frac{d}{2md-n}\right)$:

$$\left(\frac{d}{n}\right)\left(\frac{2md-n}{d}\right)\left(\frac{d}{2md-n}\right)=(-1)^{\frac{1}{2}(d-1)\cdot\frac{1}{2}(n+1)}\left(\frac{d}{2md-n}\right);$$

or, $$\left(\frac{d}{n}\right)(-1)^{\frac{1}{2}(2md-n-1)\cdot\frac{1}{2}(d-1)}=(-1)^{\frac{1}{2}(d-1)\cdot\frac{1}{2}(n+1)}\left(\frac{d}{2md-n}\right).$$

5. Multiply by $(-1)^{\frac{1}{2}(2md-n-1)\cdot\frac{1}{2}(d-1)}$ and reduce:

$$\left(\frac{d}{n}\right)=(-1)^{md(d-1)/2}\left(\frac{d}{2md-n}\right).$$

6. $\therefore\left(\frac{d}{n}\right)=(-1)^{m(d-1)/2}\left(\frac{d}{2md-n}\right)$, since $(-1)^{md(d-1)/2}=(-1)^{m(d-1)/2}$, d being an odd integer.

349. Proposed by JOSEPH A. NYBERG, Student, University of Chicago.

To show that the determinant of the nth order:

$$D_n=\begin{vmatrix} C & -1 & 0 & 0 & 0 & 0 & 0 & . & . \\ -1 & C & -1 & 0 & 0 & 0 & 0 & . & . \\ 0 & -1 & C & -1 & 0 & 0 & 0 & . & . \\ 0 & 0 & -1 & C & -1 & 0 & 0 & . & . \\ 0 & 0 & 0 & -1 & C & -1 & 0 & . & . \\ 0 & 0 & 0 & 0 & -1 & C & -1 & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{vmatrix}$$

has the value: $D_n=C^n+\sum_{r=1}^{n}(-1)^r\frac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!}C^{n-2r}$.

I. Solution by the PROPOSER.

If the determinant is expanded by the elements of the first row or column, we get:

$$D_n=C.D_{n-1}-D_{n-2},$$

where D_{n-1} and D_{n-2} are determinants similar to D_n but having, respectively, one and two fewer rows and columns. Compare

(1) $$D_n+D_{n-2}=C.D_{n-1}$$

with

(2) $$\sin(a+b)+\sin(a-b)=2\sin a\cos b.$$

Put $a=n\theta$, $b=\theta$. Then (2) becomes

$$\sin(n+1)\theta+\sin(n-1)\theta=2\sin n\theta\cos\theta.$$

Now put

(3) $$C=2\cos\theta \text{ and } D_n=c\sin(n+1)\theta,$$

where c is independent of n. To determine its value we note that, for $n=1$,

$$D_1=C=2\cos\theta\equiv\frac{\sin 2\theta}{\sin\theta}.$$

(4) $\therefore c=\csc\theta$.

As a result of (3) and (4) we can write:

$$D_n=\frac{\sin(n+1)\theta}{\sin\theta}.$$

But, from Chrystal's *Algebra*, Vol. 2, p. 252, we have:

$$(5)\quad \frac{\sin(n+1)\theta}{\sin\theta}=(2\cos\theta)^n+\sum_{r=1}^{n}(-1)^r\frac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!}(2\cos\theta)^{n-2r}$$

Using equations (5) and (3) the desired formula is obtained.

A second solution, though quite long, and therefore will only be sketched here, may be given as follows: We have

$$(1)\quad \begin{aligned} D_i&=C^i+a_{1i}\,C^{i-1}+a_{2i}\,C^{i-2}+a_{3i}\,C^{i-3}+\ldots\\ D_{i+1}&=C^{i+1}+a_{1,i+1}\,C^i+a_{2,i+1}\,C^{i-1}+a_{3,i+1}\,C^{i-2}+\ldots\\ D_{i+2}&=C^{i+2}+a_{1,i+2}\,C^{i+1}+a_{2,i+2}\,C^i+a_{3,i+2}\,C^{i-1}+\ldots \end{aligned}$$

But since $D_{i+2}=C.D_{i+1}-D_i$, we have

$$(2)\quad D_{i+2}=C^{i+2}+a_{1,i+1}\,C^{i-1}+(a_{2,i+1}-1)\,C^i+(a_{3,i+1}-a_{1,i})\,C^{i-1}+\ldots$$

Comparing (2) with the last of (3), it can be shown that $a_{jp}=0$ when j is odd. Consequently the powers of C diminish by 2.

By studying the successive coefficients of powers of C we can show by induction that the coefficient of $C^{(n-2r)}$ is

$$(-1)^r\frac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!},$$

which then gives the formula as originally stated.

II. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology.

Expanding in terms of the first column, we have the following relation connecting three determinants of the kind here considered whose orders are n, $n-1$, $n-2$:

$$D_n=CD_{n-1}-D_{n-2}.$$

Forming some of the successive values of D we find:

$$\begin{aligned} D_1&=C,\\ D_2&=C^2-1,\\ D_3&=C^3-2C,\\ D_4&=C^4-3C^2+1, \end{aligned}$$

$$D_5=C^5-4C^3+3C,$$
$$D_6=C^6-5C^4+6C^2-1,$$
$$D_7=C^7-6C^5+10C^3-4C,$$
$$D_8=C^8-7C^6+15C^4-10C^2+1.$$

It is clear that starting with D_n and reading the terms diagonally we have the expansion of $(C-1)^n$. For instance, starting with D_4 we have $C^4-4C^3+6C^2-4C+1$. Hence, reading horizontally, the 1, 2, 3, 4, etc., terms of D_n will be the 1, 2, 3, 4, etc., terms in the expansions of $(C-1)^n$, $(C-1)^{n-1}$, $(C-1)^{n-2}$, $(C-1)^{n-3}$, etc., respectively. The rth term will be the rth term of the expansion of $(C-1)^{n-r+1}$. Hence

$$D_n=C^n-(n-1)C^{n-2}+\frac{(n-2)(n-3)}{2!}C^{n-4}-\frac{(n-3)(n-4)(n-5)}{3!}C^{n-6}+\ldots$$

$$\text{or } D_n=C^n+\sum_{r=1}^{n}(-1)^r\frac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!}C^{n-2r}.$$

A very neat solution of this problem was also received from a contributor who failed to sign his name to the solution. Will contributors please note that we wish them to put their names to the solutions and to observe the order of the printed solutions, viz., put name at beginning of solution rather than at the end?

GEOMETRY.

369. Proposed by W. J. GREENSTREET, A. M., Editor, Mathematical Gazette, Stroud, England.

Prove by inversion that if two circles cut at a given angle, touch each a given circle, and pass each through the same fixed point, then shall the envelope of the points of contact be a conic.

Discussion by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Since all of the contact points lie on the fixed circles it seems probable that the desired locus is that of the second point of intersection of the variable circles. In Fig. 1, let S_1 and S_2 be the fixed circles, T_1 and T_2 the variable circles passing through the fixed point Q at the constant angle θ.

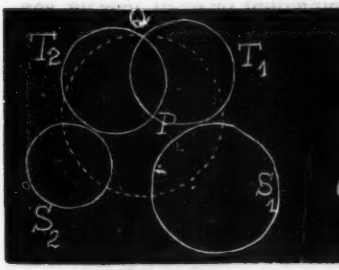


Fig. 1.

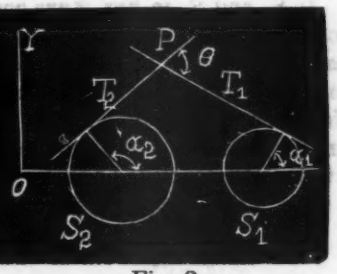


Fig. 2.

Then the locus to be found is that of the second intersection P. Through the fixed circles and Q considered as a point circle, Fig. 1, there may be passed one orthogonal circle whose center is their radical center. With Q as the center of inversion and with any convenient radius, Fig. 1 is to be inverted, thus changing T_1 and T_2 into straight lines in Fig. 2, crossing at the constant angle θ, and tangent to the new fixed circles S_1 and S_2.

The orthogonal circle mentioned above inverts into the straight line through the centers of S_1 and S_2.

The analytic work from this stage is as follows: The equations of S_1 and S_2 may be taken as

$$(x-a_1)^2+y^2=r_1^2 \text{ and } (x-a_2)^2+y^2=r_2^2,$$

hence the equations of T_1 and T_2 are

$$(x-a_1)\cos\alpha_1+y\sin\alpha_1=r_1 \text{ and } (x-a_2)\cos\alpha_2+y\sin\alpha_2=r_2,$$

in which $\alpha_1-\alpha_2=\theta$=constant, this being the angle condition. Replacing θ by 2ϕ and writing $\alpha_1=\alpha+\phi$, $\alpha_2=\alpha-\phi$, the equations of T_1 and T_2 become

$$(x-a_1)\cos(\alpha+\phi)+y\sin(\alpha+\phi)=r_1, \quad (x-a_2)\cos(\alpha-\phi)+y\sin(\alpha-\phi)=r_2.$$

Expanding the trigonometric functions and solving the two equations for $\cos\alpha$ and $\sin\alpha$ leads to the elimination of α, and gives the following for the locus of P:

$$\begin{aligned}&\{[(x-a_1)(x-a_2)+y^2]\sin\theta-y(a_1-a_2)\cos\theta\}^2\\&=r_1^2[(x-a_2)^2+y^2]+r_2^2[(x-a_1)^2+y^2]\\&-2r_1r_2\{[(x-a_1)(x-a_2)+y^2]\cos\theta+y(a_1-a_2)\sin\theta\}.\end{aligned}$$

Two special cases of the preceding equation may best be considered at this time. If S_1 and S_2 in Fig. 2 are coincident or even merely concentric, then the locus of P is a circle by elementary geometry, hence the locus of P in Fig. 1 is also a circle. This case occurs when Q, the fixed point in Fig. 1, is either of the point circles of the family of circles having a common radical axis determined by S_1 and S_2. From the general equation this result appears by placing $a_1=a_2$, and then the locus degenerates into a point circle and the circle mentioned above. Again the general equation degenerates when $r_1=r_2=0$, giving

$$\{[(x-a_1)(x-a_2)+y^2]\sin\theta-y(a_1-a_2)\cos\theta\}^2=0,$$

which is the circle, doubly counted, resulting from making S_1 and S_2 poin circles in both figures. As in the preceding case, the locus of P in Fig. 1 is also a circle. The general locus is a bicircular quartic, hence its inverse is also a bicircular quartic, thus determining the locus of P (Fig. 1) which constitutes the solution of the given problem.

NOTE. No solutions of 374 and 376 have yet been received. We shall be pleased to have our contributors take up these two problems for solution. ED. F.

CALCULUS.

300. Proposed by F. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve the differential equation obtaining the complete primitive,

$$(x^2+x^2y+2xy-y^2-y^3)dx+(y^2+xy^2+2xy-x^2-x^3)dy=0.$$

Solution by V. M. SPUNAR, Chicago, Ill., and the PROPOSER.

If F is an integrating factor of the differential equation

$$Mdx+Ndy=0,$$

we have the relation

$$\frac{F'(v)}{F(v)}=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N\frac{\partial v}{\partial x}-M\frac{\partial v}{\partial y}}=\frac{4(x-y)(x+y+1)}{N\frac{\partial v}{\partial x}-M\frac{\partial v}{\partial y}}.$$

On account of the symmetry of M and N, it is evident that v is symmetrical in x and y. Trying $v=x+y$, we have

$$\frac{F'(v)}{F(v)}=-\frac{4(x+y+1)}{(x+y)^2+2(x+y)}=-\frac{4(v+1)}{v^2+2v}.$$

Integrating, we have,

$$F=\frac{1}{v^2(v+2)^2}=\frac{1}{(x+y)^2(x+y+2)^2}.$$

Trying, also, $v=(1+x)(1+y)$, we have,

$$\frac{F'(v)}{F(v)}=-\frac{2}{x+xy+y}=-\frac{2}{v-1}.$$

Integrating,

$$F=\frac{1}{(v-1)^2}=\frac{1}{(x+xy+y)^2}.$$

If we have two integrating factors, their ratio, put equal to a constant, is the complete primitive.

Therefore, the complete primitive is

$$\frac{(x+y)^2(x+y+2)^2}{(x+xy+y)^2}=c,$$

or, extracting the square root,

$$\frac{(x+y)(x+y+2)}{x+xy+y}=c_1,$$

or, in the form,

$$\frac{x^2+y^2}{x+xy+y}=c_2.$$

301. Proposed by C. N. SCHMALL, New York City.

Show that the volume of the surface,

$$\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1, \text{ is } \frac{100\,\pi\,abc}{3.5.7.11.13}.$$

I. Solution by FRANCIS RUST, C. E., Pittsburg, Pa.

Volume is special case of Derichlet's special formula:

$$\int\int\int\ldots dx\,dy\,dz\ldots x^{l-1}.y^{m-1}.z^{n-1}\ldots$$

$$=\frac{a^l.b^m.c^n\ldots}{p.q.r\ldots}\cdot\frac{\Gamma\left(\frac{l}{p}\right).\Gamma\left(\frac{m}{q}\right).\Gamma\left(\frac{n}{r}\right)\ldots.}{\Gamma\left(1+\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+\ldots.\right)},$$

provided, $\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r+\ldots=1$, and the integration to extend over all *positive* values of the variables.

In this case,

$$V=\int\int\int dx\,dy\,dz=\frac{abc}{p.q.r}\cdot\frac{\Gamma\left[\frac{1}{p}\right].\Gamma\left[\frac{1}{q}\right].\Gamma\left[\frac{1}{r}\right]}{\Gamma\left[1+\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right]}$$

with $p=q=r=\frac{2}{5}$. The formula yields $\frac{1}{8}$ part of the total volume, which consequently is

$$V=8.\tfrac{1}{8}\tfrac{2}{4}\tfrac{5}{6}.abc.\frac{[\Gamma(\frac{5}{2})]^3}{\Gamma(\frac{7}{2})}=\frac{20.\pi}{3.7.11.13}.abc.$$

II. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, and V. M. SPUNAR, Chicago, Ill.

The solid is evidently symmetrical with respect to the axes.

Solving, we find $z=c\left[1-\left(\frac{x}{a}\right)^{\frac{3}{2}}-\left(\frac{y}{b}\right)^{\frac{3}{2}}\right]^{\frac{2}{3}}$.

When $z=0$, $y=b\left[1-\left(\frac{x}{a}\right)^{\frac{3}{2}}\right]^{\frac{2}{3}}=w$, for brevity; then

$$z=c\left[\left(\frac{w}{b}\right)^{\frac{3}{2}}-\left(\frac{y}{b}\right)^{\frac{3}{2}}\right]^{\frac{2}{3}}=\frac{c}{b}[w^{\frac{3}{2}}-y^{\frac{3}{2}}]^{\frac{2}{3}}.$$

$$V=8\int_0^a dx\int_0^w \frac{c}{b}[w^{\frac{3}{2}}-y^{\frac{3}{2}}]^{\frac{2}{3}}\,dy.$$

Let $y=w\sin^5\theta$, $dy=5w\sin^4\theta\cos\theta\,d\theta$.

$$V=\frac{40c}{b}\int_0^a dx\int_0^{\frac{1}{2}\pi} w^2\cos^6\theta\sin^4\theta\,d\theta=\frac{40c}{b}\int_0^a w^2\,dx\left[\frac{\sin^5\theta\cos^5\theta}{10}\right.$$

$$\left.+\tfrac{1}{2}\tfrac{1}{5}\tfrac{1}{6}(3\theta-\sin 4\theta+\frac{\sin 8\theta}{8})\right]_0^{\frac{1}{2}\pi}=\tfrac{6}{2}\tfrac{0}{5}\tfrac{0}{6}\frac{c}{b}\pi\int_0^a b^2\left[1-\left(\frac{x}{a}\right)^{\frac{3}{2}}\right]^5 dx$$

$$=\tfrac{6}{2}\tfrac{0}{5}\tfrac{0}{6}.\frac{256}{7.9.11.13}\pi\,abc=\frac{20\,\pi\,abc}{3.7.11.13}.$$

Also solved by J. Scheffer and the Proposer.

MECHANICS.

251. Proposed by J. G. ROSE, B. A. (Oxion), Mt. Angel College, Oregon.

ABC is a uniform triangular lamina of weight $3W$ such that $AB=2AC$. A particle of weight W is attached to it at C. Show that if the lamina be suspended from angle A, it will rest with AB and AC equally inclined to the vertical.

Solution by J. EDWARD SANDERS, U. S. Weather Bureau, Columbus, Ohio.

Draw the median AOD, AO being two thirds of AD, the vertical AP, P and D on BC, and OE and CF, perpendicular to AP. Let $AC=b$, $AB=2b$, $\angle CAB=a$, $\angle CAD=\theta$, and $\angle CAP=\phi$.

Now $BC^2=AC^2+AB^2-2AC.AB\cos\alpha=5b^2-4b^2\cos\theta$.

$\therefore CD=\frac{1}{2}b\sqrt{(5-4\cos\alpha)}$, and $4AD^2=2(AC^2+AB^2)-BC^2$

$=5b^2+4b^2\cos\alpha$.

$\therefore AD=\frac{1}{2}b\sqrt{(5+4\cos\alpha)}$, and $AO=\frac{1}{3}b\sqrt{(5+4\cos\alpha)}$.

Also $\cos\theta=\dfrac{AC^2+AD^2-CD^2}{2AC.AD}$

$$=\frac{b^2+\frac{1}{4}b^2(5+4\cos\alpha)-\frac{1}{4}b^2(5-4\cos\alpha)}{2b.\frac{1}{2}b\sqrt{(5+4\cos\alpha)}}=\frac{1+2\cos\alpha}{\sqrt{(5+4\cos\alpha)}}.$$

$$\therefore \sin\theta=\frac{2\sin\alpha}{\sqrt{(5+4\cos\alpha)}}.$$

Since the uniform weight, $3W$, acts at the centroid O, we have, by taking moments about A,

$$W.b\sin\phi=3W.\tfrac{1}{3}b\sqrt{(5+4\cos\alpha)}\sin(\theta-\phi).$$

$$\begin{aligned}\therefore \sin\phi&=\sqrt{(5+4\cos\alpha)}\sin(\theta-\phi).\\&=\sqrt{(5+4\cos\alpha)}(\sin\theta\cos\phi-\cos\theta\sin\phi)\\&=2\sin\alpha\cos\phi-(1+2\cos\alpha)\sin\phi\\&=2\sin\alpha\cos\phi-2\cos\alpha\sin\phi-\sin\phi\\&=\sin(\alpha-\phi).\end{aligned}$$

$\therefore \phi=\frac{1}{2}\alpha$.

Also solved by J. Scheffer and S. G. Barton.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

Edited by Dr. G. E. Wahlin, University of Illinois.

174. Proposed by B. KRAMER, Student, University of Pittsburg, Pittsburg, Pa.

Find a general solution of $x(x+a)=y^2$, a, x, and y being integers. Given a, required to find x to satisfy conditions.

II. Solution by E. B. ESCOTT, Ann Arbor, Mich.

The solution by Prof. F. L. Griffin is defective. His statsment that $\dfrac{n^2}{a-2n}$ is integral only when n is a multiple of $a-2n$ is not true.

For example, if $a=15$, $n=3$, $\dfrac{n^2}{a-2n}=1$, but $\dfrac{n}{a-2n}=\frac{1}{3}$.

Correct solution. Let the highest common divisor of x and $x+a$ be m. Then evidently the solution is

$$x=mr^2 \ldots (1),$$
$$x+a=ms^2 \ldots (2),$$
$$y=mrs \ldots (3).$$

From (1) and (2), $a=m(s^2-r^2)$.

Then if a is given, we can put s^2-r^2 equal to the different factors of a, and m equal to the remaining factor. s^2-r^2 may be put equal to any factor of a excepting the double of an odd number.

Example. $a=15$. If $a^2-r^2=15$, $s+r=15$ or 5, $s-r=1$ or 3,

$$\left.\begin{matrix} s=8 \\ r=7 \end{matrix}\right\} \text{ or } \left.\begin{matrix} s=4 \\ r=1 \end{matrix}\right\}, \quad m=1.$$

These give $x=49$, $y=56$, $x=1$, $y=4$.

This last solution is omitted in Professor Griffin's table owing to his incorrect analysis.

If $s^2-r^2=5$, $m=3$, $x=12$, $y=18$.

If $s^2-r^2=3$, $m=5$, $x=5$, $y=10$.

179. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve the equation in integers, $x^n+y^n+z^n+xyz=100x+10y+z$.

Solution by A. H. HOLMES, Brunswick, Maine.

Put $xyz=100x$. $\therefore yz=100$, and $z=100/y$.

$\therefore x^n+y^n+\frac{100^n}{y^n}=10y+\frac{100}{y}$; or $x^n=10y+\frac{100}{y}-y^n-\frac{100^n}{y^n}$.

For n even, $10y+\frac{100}{y}>y^n+\frac{100^n}{y^n}$.

Put $n=2$. Then $10y+\frac{100}{y}>y^2+\frac{10000}{y^2}$.

But $y=\frac{100}{z}=1, 2, 4, 5, 10, 20, 25, 50$, or 100. $\therefore n=1$ and $x=9y$.

$\therefore n=1$: $\left|\begin{matrix} x=9, 18, 36, 45, 90, 180, 225, 450, 900. \\ y=1, 2, 4, 5, 10, 20, 25, 50, 100. \\ z=100, 50, 25, 20, 10, 5, 4, 2, 1. \end{matrix}\right.$

Put $xyz=10y$. $\therefore xz=10$, $z=\frac{10}{x}$ and $x^n+y^n+\frac{10^n}{x^n}=100x+\frac{10}{x}$ or $y^n=100x+\frac{10}{x}-x^n-\frac{10^n}{x^n}$.

Put $n=1$. Then $y=99x$. But $x=10/z=1, 2, 5, 10$.

$$\therefore n=1 : \begin{vmatrix} x=1, 2, 5, 10, \\ y=99, 198, 495, 990. \\ z=10, 5, 2, 1. \end{vmatrix}$$

Put $n=2$. Then $y^2=100x+10/x-x^2-100/x^2$. $\therefore x=1$ and 10.

$$\therefore n=2 : \begin{vmatrix} x=1, 10. \\ y=3, 30. \\ z=10, 1. \end{vmatrix}$$

In neither case, when $y=1, 2, 4, 5, 10, 20, 25, 50, 100$, and $x=1, 2, 5, 10$, can $n=3$.

180. Proposed by A. H. HOLMES, Brunswick, Maine.

Find integral values for x and y in the following: $96x-96y+21=\square$.

Solution by the PROPOSER.

If we solve the equation

$$(1) \quad (24y+z)^2=25(24x+z-1),$$

for z we find

$$z=\frac{25-48y\pm 5\sqrt{(96x-96y+21)}}{2}.$$

Since the coefficient of z^2 in (1) is unity, if for a pair of values of x and y, $96x-96y+21$ is a square, the corresponding value of z must be a rational integer. But no set of integral rational values of x, y, z will satisfy (1), for if the value of z is odd, the left hand member of (1) is odd and the right hand member is even, and vice versa for z even. Hence there are no integral values for x and y such that the given expression is a square.

☞ Problems and solutions for this department should be sent to Dr. Wahlin, Urbana, Ill.

PROBLEMS FOR SOLUTION.

ALGEBRA.

353. Proposed by DANIEL KRETH, Oxford, Iowa.

Divide 2940 into two such factors that the square of one factor minus 21 will equal three times the other factor. 21 140

354. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

To find x in the following equation:

$$0.002\{6x-20[(1.05)^x-1]\}=0.012\{21[(1.05)^{x-1}-1]-(x-1)\}.$$

GEOMETRY.

384. Proposed by S. LEFSEHETZ, Clark University.

Let ABC be a triangle, O a circle tangent to its three sides, T a variable tangent of O, which cuts the sides BC, CA, AB in a, b, c. Oa', Ob', Oc' the perpendiculars in O to Oa, Ob, Oc, cutting, respectively, T in points a', b', c'. Prove that Aa', Bb', Cc' meet in a point t, and find the locus of t when T varies. Purely geometrical proofs wanted.

385. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Given a triangle ABC, find the radius of a circle touching two of its sides and a line parallel to the third, at a distance $d=u+2r$.

386. Proposed by DANIEL KRETH, Oxford, Iowa.

Construct the triangle, having given, the vertical angle, the sum of the three sides, and the perpendicular.

CALCULUS.

308. Proposed by C. N. SCHMALL, New York City.

Prove, by calculus, that of all isoperimetric triangles, the equilateral has the greatest area.

309. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology.

In practical problems involving maxima and minima, it is really the greatest and least values of the function which are desired. Show why we can assume that the maximum is the greatest value and the minimum the least value under the conditions.

310. Proposed by C. N. SCHMALL, New York City.

Evaluate $\int_0^\pi \frac{dx}{1-2a\cos x+a^2}$. Edwards' *Integral Calculus for Beginners*, page 131, ex. 9, (iii). The answer given is $\frac{\pi}{1-a^2}$. Is this a complete answer to the question?

MECHANICS.

260. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

To the ends of a fine inextensible string, length $2l$, are attached to equal, smooth, spherical, equally elastic (e) particles. At first the middle point of the string touches a rigid, fixed, circular rim, radius a, and the particles are $2l$ apart. They are now projected with equal velocities perpendicular to the string and curl around the rim. If l is greater than πa, find the condition that the particles will move after collision along tangents to the rim, the whole motion being on a smooth horizontal plane.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

183. Proposed by M. T. GOODRICH, Dixfield, Maine.

Show what relation must exist between the quantities A, B, and C, in the harmonic ratio $\frac{AB}{(A+B+C)(-C)}=-1$, so that they will be real positive integers.

NOTES AND NEWS.

Professor E. B. Van Vleck is again in residence at the University of Wisconsin, not at the University of Minnesota, as inadvertantly announced last month. S.

Editor Slaught left the University of Chicago March 21st, to make a trip through the South, in the interest of the University. He will make a number of speeches before college students and educational organizations and will reach home about the first week in April. F.

Dr. A. M. Hiltebeitel has been appointed instructor in mathematics at the University of Pennsylvania. Dr. H. B. Smith has been appointed instructor in the same department for the ensuing term, to fill the vacancy caused by the temporary absence of Professor Evans. S.

The Annals of Mathematics will be taken over by Princeton University after the completion of the present volume. This journal was founded at the University of Virginia, under whose auspices it was published until it was taken over by Harvard University a few years ago. S.

A splendid edition of Halsted's Rational Geometry, in French, is issuing from the presses of Gauthier-Villars, the famous publishing house of Paris. In the history of geometry this is a fine compliment to Dr. Halsted, as we believe no geometry of the Western Hemisphere has ever before been so honored. F.

The syllabus on calculus, forming the fourth section of the report of the committee on mathematics for students of engineering, of which Professor Huntington, of Harvard University, is chairman, will appear in the forthcoming number of the *Bulletin* of the Society for the Promotion of Engineering Education. The committee presented a preliminary report at the joint meeting of engineers and mathematicians at Minneapolis last December, and it is hoped to present a completed report at the coming summer meeting of the Society for the Promotion of Engineering Education. S.

Three important reports of American Committees under the International Commission on the teaching of mathematics, have been printed in the *Bulletin of the American Mathematical Society*, namely: (1) On the preparation of instruction for colleges and universities, in Volume XVII, No. 2; (2) On university courses in mathematics and the Master's degree, in Volume XVII, No. 5; (3) Preparation for research and the Doctor's degree in mathematics, in Volume XVII, No. 6. These reports and all the others of the American Committee will be published later by the Department of Education at Washington. S.

The April meeting of the American Mathematical Society will be held at the University of Chicago, on Friday and Saturday, April 28-29. At this meeting Professor Maxime Bôcher will deliver his Presidential address, the provisional title of which is: "Charles Sturm's Published and Unpublished Work on Differential and Algebraic Equations." The announcement states that: "Except for the summer meetings, this is the first gathering of the whole society that has been held since 1896, and it is hoped to make it a memorable occasion. The full attendance of the Chicago Section should be reinforced by large delegations from both East and West." S.

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THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XVIII. APRIL, 1911. NO. 4.

RITHMOMACHIA, THE GREAT MEDIEVAL NUMBER GAME.

By DAVID EUGENE SMITH and CLARA C. EATON.

When the subject of number games shall be adequately treated, and the long and interesting story is told of how the world has learned to handle the smaller numbers quite as much through play as through commerce, the climax will probably be found in the chapter relating to the Battle of Numbers, the Rithmomachia of the Middle Ages. For here was a tournament worthy of intellectual foes, a play that outranked chess as much as chess surpasses mere dicing, and a game that was by its very nature closed to all save selected minds that had been trained in the Boethian arithmetic, the Latinized Nicomachus, the last great effort in the Pythagorean philosophy of numbers.

But when this story comes to be told the one who relates it will have no easy task, and the object of this paper is rather to set forth the problem than to solve it. For although we have manuscripts of three writers of the eleventh century, two of the twelfth, one of the thirteenth, and Bradwardin's work of the fourteenth,* and although we have several printed treatises on the subject,† we know practically nothing of the origin of the game. We only know that the medieval writers attributed it to Pythagoras, that no trace of it has been discovered in Greek literature, and that no mention of it has been found before the time of Hermannus Contractus (1013-1054). The name, which appears in a variety of forms,‡ points to a Greek origin, the more so because Greek was little known at the time when the game first appears in literature. Based as it is upon the Greek theory of numbers,

* There is a twelfth century manuscript of Hermannus Contractus (1013-1054) at Paris, and others of later date in various libraries. Wappler has published this, and also a treatise by Asilo (before 1077), with part of an anonymous one of the twelfth century. Odo also wrote on the subject in the eleventh century. Peiper has edited Fortolfus's work of the twelfth century. Several other manuscripts are known. Consult Wappler, in the *Zeitschrift für Mathematik und Physik*, Vol. 37, p. 1 (1892), and Peiper in the *Abhandlungen*, Vol. 3 (1880).

† We have made free use of the brief description given by Jacobus Faber Stapulensis (1496), possibly from Shirewood's manuscript, and the works of Boissière (French edition 1554, Latin edition 1556) and Baroszi (1572). Abraham Riese (1562) published Asilo's manuscript.

‡ Correctly, Rithmomachia, but also in such incorrect forms as Rythmomachia (Battle of Rhythms), Rithmimachia, Rythmimachia, etc.